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Satake compactifications of arithmetic quotients

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Introduction

Throughout this paper, let

$G =$ the $\mathbb{R}$-rational points on a semi-simple group defined over $\mathbb{R}$
$K =$ a maximal compact subgroup of G
$X =$ the corresponding symmetric space, which may be identified with G/K

For the moment suppose that G is defined over $\mathbb{Q}$ and that Γ is an arithmetic subgroup.

In the paper [Satake 1960a], Satake showed how to associate G-covariant compactifications of X to irreducible representations of G, and in the subsequent paper [Satake 1960b] he showed how some of these could give rise to compactifications of the arithmetic quotient $\Gamma\backslash X$. He gave several examples of his procedure, but it was not clear which compactifications of X could be used for this purpose—i.e., which of them were what will be defined in this paper to be **geometrically rational**. Shortly afterwards it was shown in [Borel 1962] that those associated to $\mathbb{Q}$-rational representations of G were geometrically rational. But it was also known from examples among what are now known as Baily-Borel compactifications that the rationality of π was not actually necessary. It seems to have remained an open problem, however, to formulate a relatively simple necessary and sufficient criterion for geometric rationality. This paper is an attempt to do this.

The literature on compactifications dealing with these questions, outside the early papers already mentioned, is rather sparse. One exception is the book [Ash *et al.* 1975], where the case by case argument of [Baily-Borel 1966] was replaced by a more direct proof of geometric rationality. Another exception is [Zucker 1986b], where the general question of geometric rationality is broached perhaps for the first time since the early papers of Satake and Borel. Indeed, as will be apparent, this paper was to a large extent inspired by Zucker's. In order to avoid confusion, however, I should point out that not only is the logic of the discussion in §3 of his paper rather unclear, but that in addition there is an actual error in his Proposition (3.3). (This does not, however, affect the remainder of the paper. Roughly speaking, the Corollary to the Proposition should replace his Assumption 1 as a hypothesis.)

I take pleasure in thanking Satake for his part in arranging my present one-year visit to Tohoku University, during which this paper was written.

Details will appear in a subsequent paper, in which compactifications associated to reducible representations will be treated as well.

1. Compactifications of X

Let (π, V) be an irreducible finite-dimensional complex representation of G possessing a vector v such that K is the stabilizer of the line $[v]$ through v. Such a representation, or rather the triple (π, V, v), I call **projectively spherical**. In Satake's paper, he starts out with an arbitrary irreducible representation of G, and then considers the associated representation on the space of Hermitian forms on V, which contains a positive definite form invariant under K. The representation I would look at would be the one generated by this form.

1.1. Proposition. *Any irreducible projectively spherical representation (π, V, v) of G is strongly rational over $\mathbb{R}$.*

That is to say that both V and v may be chosen real, and the stabilizer of the highest weight vector is a real parabolic subgroup of G. Several proofs must be possible, but the one I have in mind is from representation theory: any irreducible finite-dimension of G can be embedded into a principal series representation, that is to say a G-space of functions

$$\mathrm{Ind}(\chi \mid P, G) = \{f \in C^\infty(G, U) \mid f(pg) = \chi(p) f(g) \text{ for all } p \in P, g \in G\}$$

where P is a minimal real parabolic subgroup of G and (χ, U) is a finite-dimensional representation of M_P. The Iwasawa decomposition $G = PK$ implies that the restriction of this to K is

$$\mathrm{Ind}(\chi \mid K \cap P, K).$$

If π is projectively spherical, Frobenius reciprocity implies that χ must be a character of order one or two.

If the kernel of π is contained in K, and in particular if π is essentially faithful, the map $g \mapsto [\pi(g)v]$ identifies $X = G/K$ with a subspace of the projective space $\mathbb{P}(V)$. Let $\overline{X} = \overline{X}_\pi$ be the closure of X in $\mathbb{P}(V)$.

Fix a minimal real parabolic subgroup $P_{\emptyset,\mathbb{R}}$. Let A be the connected component of a maximal $\mathbb{R}$-split torus in $P_{\emptyset,\mathbb{R}}$ which is stable under the Cartan involution determined by K. Let Y be the orbit of $[v]$ under A.

1.2. Lemma. *Every G-orbit in $\overline{X}$ meets $\overline{Y}$.*

This follows immediately from the Cartan decomposition $G = KAK$ and the compactness of K.

What does $\overline{Y}$ look like? Suppose

$$v = \sum_{\alpha \in \Sigma(v)} v_\alpha$$

to be the decomposition of v into eigenvectors of A. Define $\Omega(v)$ to be the convex hull in $X^*_{\mathbb{R}}(A)$ of the set $\Sigma(v)$ of characters appearing with non-zero coefficient in this decomposition, and $\mathfrak{V}(v)$ to be its extremal vertices. Then $\Omega(v)$ is a closed, bounded, convex polyhedron of dimension equal to the dimension of A. If F is any face of $\Omega(v)$, then define v_F to be the sum of the the v_α for α in F—the A-projection of v onto the space spanned by the eigenvectors corresponding to characters in F—and Y_F the A-orbit of $[v_F]$ in $\mathbb{P}(V)$. If λ in $X_*(A)$ is a one-parameter subgroup of A, then for any t in $\mathbb{R}^\times$ we have

$$\pi(\lambda(t))v = \sum t^{\langle \lambda, \alpha \rangle} v_\alpha.$$

If F is the interior of the closed face of $\Omega(v)$ where the linear functional λ achieves its maximum value $\mu(\lambda)$, then

$$\pi(\lambda(t))v = t^{-\mu(\lambda)} \sum t^{\mu(\lambda) - \langle \lambda, \alpha \rangle} v_\alpha,$$

and

$$\lim_{t \to \infty} [\pi(\lambda(t))v] = \sum_{\alpha \in F} v_\alpha = [v_F].$$

Hence the closure of Y contains Y_F. A standard argument from [Kempf *et al.* 1973] implies further that, conversely, the union of all the Y_F is compact. In other words:

1.3. Lemma. *The closure of the A-orbit of $[v]$ in $\mathbb{P}(V)$ is the union of the orbits of the $[v_F]$ as F varies over the faces of the convex hull of the eigencharacters of v. The orbit Y_F is contained in the closure of the orbit Y_E if and only if F is contained in the closure of E.*

The reason that $\Omega(v)$ is simple to describe is this:

1.4. Proposition. *In these circumstances, the extremal weights of v are the same as the extremal weights of V.*

One way to show this is to apply representation theory again, since there is a well known explicit formula for the coefficients of the highest weights of v.

How are the faces of $\Omega(V)$ parametrized? Fix a minimal parabolic subgroup $P_{\emptyset,\mathbb{R}}$ containing A, let W be the Weyl group, and let C be the corresponding positive Weyl chamber in root space. Since $\Omega(V)$ is Weyl-invariant, it suffices to describe those faces that meet the positive Weyl chamber C. For this I need some notions introduced apparently by Satake. Let $\Delta_{\mathbb{R}}$ be the basis for the positive real roots determined by $P_{\emptyset,\mathbb{R}}$, and let γ be the highest weight of π. Set

$$\Delta_{\mathbb{R},\pi} = \text{the roots in } \Delta_{\mathbb{R}} \text{ which are not orthogonal to } \gamma$$

Call a subset of $\Delta_{\mathbb{R}}$ **π-connected** if each of its components in the Dynkin diagram contains an element of $\Delta_{\mathbb{R},\pi}$. If Θ is a subset of $\Delta_{\mathbb{R}}$ I follow Zucker's notation and set

$$\begin{aligned} \kappa(\Theta) &= \text{the largest } \pi\text{-connected subset of } \Theta \\ \zeta(\Theta) &= \text{the orthogonal complement of } \kappa(\Theta)\text{—i.e., the elements of } \Delta_{\mathbb{R}} \text{ not edge-} \\ &\qquad \text{linked to } \kappa(\Theta) \text{ in the Dynkin diagram} \\ \omega(\Theta) &= \kappa(\Theta) \cup \zeta(\Theta). \end{aligned}$$

A subset Θ is called **π-saturated** if it $\Theta = \omega(\Theta)$. Let W_Θ be the subgroup of W generated by reflections s_α corresponding to elements α in Θ. Of course $s_\alpha \gamma = \gamma$ for α in $\zeta(\Delta_{\mathbb{R}})$, hence the orbit of γ under W_Θ is the same as its orbit under $W_{\kappa(\Theta)}$. In fact:

1.5. Proposition. *The map taking Θ to the convex hull of its orbit under W_Θ is an inclusion-preserving bijection between the π-connected subsets of $\Delta_{\mathbb{R}}$ and the faces of the convex hull of the W-orbit of γ meeting C.*

The **standard** parabolic subgroups are those containing $P_{\emptyset,\mathbb{R}}$. They are parametrized by subsets of $\Delta_{\mathbb{R}}$, so that the split centre A_Θ of the reductive component of the parabolic subgroup P_Θ is isomorphic to the intersection of kernels of α in Θ. Recall further that for any Θ the subset A_Θ^{++} is the set of all a in A_Θ with $\alpha(a) > 1$ for all the α in $\Delta_{\mathbb{R}}$ not in Θ. The earlier calculation shows:

1.6. Proposition. *If the image under $\lambda \in X_{*,\mathbb{R}}(A)$ lies in A_Θ^{++} and F is the face corresponding to $\omega(\theta)$ then $\lim_{t\to\infty}[\pi(\lambda(t))v]$ lies in Y_F.*

In these circumstances, define P_F to be the parabolic subgroup corresponding to Θ in the usual parametrization of parabolic subgroups containing a given minimal one.

1.7. Proposition. *The parabolic subgroup P_F is the stabilizer of the subspace V_F, which is equal to* $\mathrm{Fix}(N_F)$. *If Q is the parabolic subgroup of G corresponding to the subset Φ, then* $\mathrm{Fix}(N_Q) = V_F$ *if and only if* $\kappa(\Theta) \subseteq \Phi \subseteq \Theta$.

Define L_F to be the reductive subgroup of M_F isogenous to the product of A_F, the semi-simple factor of M_F whose real Dynkin diagram is that spanned by $\zeta(\Theta)$ and any compact factors of M_F. Let G_F be the semi-simple quotient M_F/L_F.

1.8. Proposition. *The group L_F is the projective kernel of the representation of M_F on V_F.*

Let X_F be the orbit of $[v_F]$ under G_F, the **boundary component** corresponding to F. It is isomorphic to the symmetric space of G_F.

1.9. Proposition. *The orbit of $[v_F]$ under G is isomorphic to the fibre product of G and X_F with respect to P_F.*

The boundary components of $\overline{X}$ therefore correspond exactly to the π-saturated parabolic subgroups of G, by which I mean those conjugate to a standard parabolic subgroup corresponding to a π-saturated subset of $\Delta_{\mathbb{R}}$.

The isogeny class of G is not determined by the Dynkin diagram of its real roots, but rather by a certain collection of data on the Dynkin diagram of its complex roots $\Delta_{\mathbb{C}}$, which is a special case of a Galois index, whose definition I recall from [Borel-Tits 1965: i]n some generality.

Suppose k to be any subfield of $\mathbb{C}$, and suppose G to be defined over k. Choose a torus S in G, maximally split over k, and let T be a maximal torus in the complexification $G(\mathbb{C})$ of G. Fix a Borel subgroup B of $G(\mathbb{C})$ containing T, and let $\Delta_{\mathbb{C}}$ be a basis for the positive roots of $(G(\mathbb{C}), T)$ determined by the choice of B. The maximal proper parabolic subgroups of $G(\mathbb{C})$ containing T are parametrized by the maximal proper subsets of $\Delta_{\mathbb{C}}$, hence essentially by elements of $\Delta_{\mathbb{C}}$. If σ is any automorphism of $\mathbb{C}/k$ and Q is any parabolic subgroup of G containing B, then there will exist a unique G-conjugate of its Galois-conjugate Q^σ containing B. Using the parametrization of the maximal parabolic subgroups by elements of $\Delta_{\mathbb{C}}$, we have therefore a homomorphism, which Borel and Tits call $*$, from the automorphism group $\mathrm{Aut}(\mathbb{C}/k)$ to the automorphism group of the complex Dynkin diagram. This $*$-action of $\mathrm{Aut}(\mathbb{C}/k)$ on the Dynkin diagram turns out not to depend on any of the choices made. This homomorphism factors through the Galois group of the algebraic closure of k in $\mathbb{C}$.

If Q is a minimal parabolic subgroup of G_k containing S, then its complexification will be a parabolic subgroup of $G(\mathbb{C})$ containing T, and we may as well choose P to be contained in Q. Restriction of roots to S determines a surjection $\rho_{\mathbb{C}/k}$ from $\Delta_{\mathbb{C}}$ to $\Delta_k \cup \{0\}$. The **anisotropic kernel** is normally the isomorphism class of the Levi component of a minimal k-rational parabolic subgroup, but I mean by it the set $\Delta^0_{\mathbb{C}/k}$, which is $\rho^{-1}_{\mathbb{C}/k}(0)$. The group $\mathrm{Aut}\,\mathbb{C}/k$ takes this subset into itself, hence its complement. The orbits in the complement are exactly the inverse images $\rho^{-1}_{\mathbb{C}/k}(\alpha)$ for α in Δ_k. For a subset $\Theta \subseteq \Delta_k$ define $\epsilon_{\mathbb{C}/k}(\Theta)$ to be $\rho^{-1}(\Theta) \cup \{0\}$.

Define $\Delta_{k,\pi}$ to be the those roots in Δ_k which are not orthogonal to the highest weight of π. It follows from a variant of Proposition 1.7 that α lies in $\Delta_{k,\pi}$ if and only if it is the restriction of a root α in $\Delta_{\mathbb{C}}$ which either lies in $\Delta_{\mathbb{C},\pi}$ or is connected to $\kappa^0_{\mathbb{C}/k} = \kappa(\Delta^0_{\mathbb{C}/k})$ by a single edge. I call the set of such roots in $\Delta_{\mathbb{C}}$ the **extended** $\Delta_{\mathbb{C},\pi}$.

Suppose $k = \mathbb{R}$. The index is usually coded into a scheme for marking the complex Dynkin diagram, black circles for elements of $\Delta^0_{\mathbb{C}/\mathbb{R}}$—which are the compact roots—white for the complement, two white circles connected by a double ended arrow if exchanged by complex conjugation. (See for example the pictures in [warner 1972: pp. 30–32].)

Define $\Delta_{\pi,\mathbb{C}}$ to be the inverse image in $\Delta_{\mathbb{C}}$ of $\Delta_{\pi,\mathbb{R}}$. By Proposition 1.1 this is the same as the complement in $\Delta_{\mathbb{C}}$ of the roots orthogonal to a highest weight vector of π. If F corresponds to the π-saturated set $\Theta = \zeta \cup \kappa$ then the Galois index of L_F is the induced index on $\epsilon_{\mathbb{C}/\mathbb{R}}(\zeta)$.

2. Compactifications of arithmetic quotients

Suppose now that G is defined over $\mathbb{Q}$. If P is a minimal $\mathbb{Q}$-rational parabolic subgroup, Ω is a compact subset of P, and $T > 0$, then the Siegel set associated to this data is the image of

$$\mathfrak{S}(P, \Omega, T) = \Omega A_{\mathbb{Q}}^{++}(T) K$$

in X, where $A_{\mathbb{Q}}$ is the maximal $\mathbb{Q}$-split torus contained in P, stable under the Cartan involution determined by K, and

$$A_{\mathbb{Q}}^{++}(T) = \{a \in A_{\mathbb{Q}} \mid \delta_Q(a) > T \text{ for all parabolic subgroups } Q \text{ containing } P\}.$$

Here δ_Q is the modulus character of Q. The main result of reduction theory is that the arithmetic quotient $\Gamma \backslash X$ is covered by a finite number of Siegel sets (one in fact for each of the finite set of Γ-conjugacy classes of minimal $\mathbb{Q}$-rational parabolic subgroups).

2.1. Proposition. *If $\mathfrak{S}$ is a Siegel set in X, then its closure in $\overline{X}$ meets just those boundary components met by the closure of $A_{\mathbb{Q}}^{++}(T)$. More precisely, if we choose $P_{\emptyset,\mathbb{R}}$ to be a minimal real parabolic subgroup contained in $P_{\mathbb{Q}}$ and obtain therefore a parametrization of all real parabolic subgroups containing $P_{\emptyset,\mathbb{R}}$, then the boundary components met are those corresponding to the subsets $\omega(\epsilon(\Theta))$, where Θ is a π-saturated subset of $\Delta_{\mathbb{Q}}$.*

I call the Satake compactification **geometrically rational** if both of the following conditions (GR1) and (GR2) hold:

(GR1) Each boundary component intersecting the closure of a Siegel set in X has as its stabilizer a $\mathbb{Q}$-rational parabolic subgroup of G.

Under these circumstances it is easy to see that, conversely, any boundary component with $\mathbb{Q}$-rational stabilizer meets the closure of some Siegel set. I will call these the **rational boundary components**.

(GR2) If P is the stabilizer of a rational boundary component X_P then the link group L_P is isogenous to the product of a rational group and a compact one.

When these conditions hold, define the space X^* to be the union of all the rational boundary components (including X itself). The conditions guarantee exactly that each of the rational boundary components is itself the symmetric space of some semi-simple group defined over $\mathbb{Q}$ (the group associated to X_P will be the product of G_P and a compact factor). Assign to X^* the topology defined by the condition that a set is open if and only if its intersection with every Siegel set in every boundary component is open.

It follows from the main result of [Satake 1960b] combined with one of the main results of [Borel 1962] that Γ acts discretely on X^*, with $\Gamma\backslash X^*$ compact and Hausdorff.

The obvious necessary and sufficient condition for (GR1) to hold is that for all π-saturated subsets Θ of $\Delta_{\mathbb{Q}}$, the subset $\epsilon_{\mathbb{C}/\mathbb{Q}}(\Theta)$ be π-saturated. Verifying this, however, is unnecessarily complicated. Consider this condition with $\Theta = \emptyset$. Since $\epsilon(\emptyset)$ is just $\Delta^0_{\mathbb{C}/\mathbb{Q}}$, $\omega(\epsilon(\emptyset))$ is the union of $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ and those roots outside $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ which are not in $\Delta_{\mathbb{C},\pi}$ and also not connected to κ^0 by a single edge. Therefore in order for the condition to hold for $\emptyset$ it is necessary and sufficient that the set of roots in $\Delta_{\mathbb{C}/\mathbb{Q}} - \Delta^0_{\mathbb{C}/\mathbb{Q}}$ which are either in $\Delta_{\mathbb{C},\pi}$ or connected to $\kappa^0_{\mathbb{C}/\mathbb{Q}}$ by a single edge be Galois-invariant. As mentioned before, I call this last subset of $\Delta_{\mathbb{C}}$ the extended $\Delta_{\mathbb{C},\pi}$. In fact this condition turns out to be sufficient in general:

2.2. Proposition. *Condition (GR1) holds if and only the complement of the anisotropic kernel $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ in the extended $\Delta_{\mathbb{C},\pi}$ is Galois invariant.*

If G is quasi-split over $\mathbb{Q}$, then the anisotropic kernel is empty. In this case the compactification is geometrically rational if and only if $\Delta_{\mathbb{C},\pi}$ is Galois invariant.

Now assume that (GR1) holds. Consider condition (GR2) for the minimal parabolic subgroup—i.e., again look at the simplest case where $\Theta = \emptyset$. Then G_F in this case has Dynkin diagram equal to κ^0. In order for (GR2) to hold, this must be Galois invariant up to some compact factors. In other words: the Galois group will permute the connected components of $\Delta^0_{\mathbb{C}/\mathbb{Q}}$. Those connected components which possess an element of $\Delta_{\mathbb{C},\pi}$ make up κ^0. Some of these may be transformed by the Galois group into some components which are not in κ^0. In order for (GR2) to hold, all of these last must comprise only compact roots. Again, this turns out to be necessary and sufficient for the general validity of (GR2):

2.3. Proposition. *Assuming that (GR1) holds, then (GR2) holds if and only if the Galois orbit of $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is contained in the union of $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ and the subset of compact roots.*

This result may be applied easily to see that every Baily-Borel compactification is geometrically rational. More generally it allows a proof which is straightforward (admittedly not elegant) of this:

2.4. Proposition. *If the Satake compactification $X \hookrightarrow \overline{X}$ has the property that for every boundary component X_P of the symmetric space X the rank of G_P is the same as* *that of its maximal compact subgroup, then it is geometrically rational.*

The real symmetric spaces with this property are listed at the end of []. The hypothesis is a bit odd in view of the fact that not all of the conjugacy classes of real boundary components occur as rational boundary components. At any rate, it would be interesting to construct in all these cases an interpretation of the structure of X which allowed one to prove this directly, in the style of the treatment of the Hermitian symmetric case in [Ash *et al.* 1975]. When the group is absolutely simple over $\mathbb{Q}$ the proof of this proposition case by case is easy, since there are so few automorphisms of Dynkin diagrams. The case when G is obtained by restriction of ground field is not much more complicated.

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